Matemáticas VII. Preparaduría

Semana #02.

f es de soporte compacto $\iff f(x) \equiv 0$ fuera de algún intervalo.

* DISTRIBUCIONES.

→ Definición general. $\langle f|\varphi\rangle_{(x)} = \int_{-\infty}^{\infty} f(x)\varphi(x)\,\partial x$; $\varphi \in \mathcal{D}(\mathbb{R}) \wedge f \in L^1(\mathbb{R})$.

→ Propiedades: $\lambda; \mu; \sigma \in \mathbb{C}$ (Constantes). $\wedge\ \varphi \wedge \Psi \in \mathcal{D}(\mathbb{R})$.

- $\langle f+g|\varphi\rangle_{(x)} = \langle f|\varphi\rangle_{(x)} + \langle g|\varphi\rangle_{(x)}$.
- $\langle f+g|\varphi+\Psi\rangle_{(x)} = \langle f+g|\varphi\rangle_{(x)} + \langle f+g|\Psi\rangle_{(x)}$
- $\langle \lambda f|\varphi\rangle_{(x)} = \lambda\langle f|\varphi\rangle_{(x)}$.
- $\langle f+g|\mu\varphi\rangle_{(x)} = \mu\langle f+g|\varphi\rangle_{(x)}$.

IMPORTANTE.

$\langle \delta_a(x)|\varphi(x)\rangle = \langle \delta(x-a)|\varphi(x)\rangle = \langle \delta_a|\varphi\rangle_{(x)} = \varphi(a)$

* DERIVADAS GENERALIZADAS (O DISTRIBUCIONALES)

– En forma general $\left\langle \frac{\partial^n f(x)}{\partial x^n} \middle| \varphi(x) \right\rangle = (-1)^n \left\langle f(x) \middle| \frac{\partial^n \varphi(x)}{\partial x} \right\rangle$

→ Definición

$$\langle f^{(n)}_{gen}(x)|\varphi(x)\rangle = (-1)^n \int_{-\infty}^{\infty} f(x)\varphi^{(n)}_{(x)}\,\partial x \quad ; \varphi \in \mathcal{D}(\mathbb{R}).$$

→ Propiedades:

- $\langle (\sigma f)'|\varphi\rangle_{(x)} = \langle \sigma' f + \sigma f'|\varphi\rangle_{(x)}$
- $\langle (\alpha f + G)'_{gen}|\varphi\rangle_{(x)} = \langle \alpha f'_{gen} + G'_{gen}|\varphi\rangle_{(x)} \Rightarrow (\alpha f + G)'_{gen} = \alpha f'_{gen} + G'_{gen}$
- Regla de Leibnitz $\rightarrow (gf)'_{gen} = g'_{cl}\cdot f + g\cdot f'_{gen}$.

IMPORTANTE.

$\langle \delta_a^{(n)}|\varphi\rangle_{(x)} = (-1)^n\langle \delta_a|\varphi^{(n)}\rangle = (-1)^n\varphi^{(n)}(a)$; $\varphi \in \mathcal{D}(\mathbb{R})$.

* Integrales Importantes.

$$\Rightarrow \int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \nexists \text{ sí } p \leq 1. \\ \frac{1}{p-1} \;\exists \text{ sí } p > 1. \end{cases} \quad ; \quad \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$$

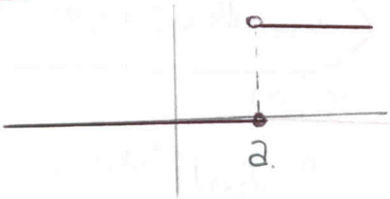
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\text{SEN}(x)}{x}\, dx = \pi \quad ; \quad \Rightarrow \int_0^{\infty} e^{-zt}\, dt = \frac{1}{z} \; ; \text{ sí } \operatorname{Re}(z) > 0. \quad z \in \mathbb{C}.$$

$\Rightarrow$ Para $\varepsilon > 0$. $\displaystyle\int_0^{\infty} e^{-\varepsilon x}\,\text{SEN}(x)\,dx = \frac{1}{\varepsilon^2+1}$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

* FUNCIONES IMPORTANTES.

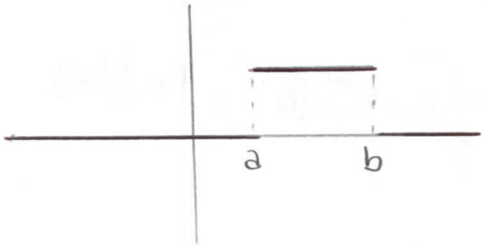
* Función Heaviside.



$$H(x-a) = H_a(x) = \begin{cases} 1 \; ; & x > a \\ 0 \; ; & x \leq a \end{cases}$$

$a \in \mathbb{R}$.

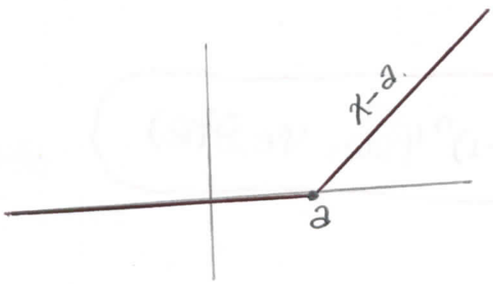
* Función Pulso.



$a \wedge b \in \mathbb{R} \wedge a < b$.

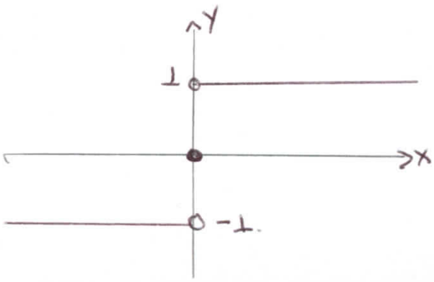
$$\Rightarrow f_{pu}(x) = \mathbb{1}_{[a,b]}(x) = H_a(x) - H_b(x) = H(x-a) - H(x-b) = \begin{cases} 1 \; ; & x \in [a;b] \\ 0 \; ; & x \notin [a;b] \end{cases}$$

* Función Rampa.



$$f(x) = (x-a)H_a(x) = (x-a)H(x-a) = \begin{cases} x-a \; ; & \forall x \geq a \\ 0 \; ; & \forall x < a \end{cases}$$

* Función SIGNO



$$f(x) = \text{sign}(x) = \text{sgn}(x) = \begin{cases} 1 \text{ sí } x > 0 \\ 0 \text{ sí } x = 0 \\ -1 \text{ sí } x < 0. \end{cases}$$

Ejercicios

1) (Parcial ENE-MAR 2012) 10 Ptos.

Expresar la función generalizada $F(x) = (\operatorname{SEN}(x))\left((\operatorname{sen}(3x))\,\delta'(x)\right)'$ como una combinación lineal de $\delta(x)$; $\delta'(x)$ y $\delta''(x)$.

Solución

Aplicando $\varphi(x)$ una función de prueba.

$$\langle (\operatorname{sen}(x))(\operatorname{sen}(3x)\,\delta'(x))' \mid \varphi(x)\rangle = \langle (\operatorname{sen}(3x)\,\delta'(x))' \mid \operatorname{sen}(x)\,\varphi(x)\rangle$$

$$= -\langle \operatorname{sen}(3x)\,\delta'(x) \mid (\operatorname{sen}(x)\,\varphi(x))'\rangle$$

$$= -\langle \delta'(x) \mid \operatorname{sen}(3x)(\operatorname{sen}(x)\,\varphi(x))'\rangle$$

$$= \langle \delta(x) \mid (\operatorname{sen}(3x)(\operatorname{sen}(x)\,\varphi(x))')'\rangle$$

$$\boxed{\langle \delta(x) \mid [\operatorname{sen}(3x)(\operatorname{sen}(x)\,\varphi(x))']'\rangle}$$

$$[\operatorname{sen}(3x)(\operatorname{sen}(x)\,\varphi(x))']' = (\operatorname{sen}(3x))'(\operatorname{sen}(x)\,\varphi(x))' + \operatorname{sen}(3x)\left[(\operatorname{sen}(x)\,\varphi(x))'\right]'$$

$$\boxed{(\operatorname{sen}(3x))' = 3\cos(3x)}$$

$$\boxed{(\operatorname{sen}(x)\,\varphi(x))' = \cos(x)\,\varphi(x) + \operatorname{sen}(x)\,\varphi'(x)}$$

$$[(\operatorname{sen}(x)\,\varphi(x))']' = (\operatorname{sen}(x)\,\varphi(x))'' = (\cos(x)\,\varphi(x) + \operatorname{sen}(x)\,\varphi'(x))'$$

$$= -\operatorname{sen}(x)\,\varphi(x) + \cos(x)\,\varphi'(x) + \cos(x)\,\varphi'(x) + \operatorname{sen}(x)\,\varphi''(x)$$

$$\boxed{[(\operatorname{sen}(x)\,\varphi(x))']' = -\operatorname{sen}(x)\,\varphi(x) + 2\cos(x)\,\varphi'(x) + \operatorname{sen}(x)\,\varphi''(x)}$$

$$[\operatorname{sen}(3x)(\operatorname{sen}(x)\,\varphi(x))']' = 3\cos(3x)\left[\cos(x)\,\varphi(x) + \operatorname{sen}(x)\,\varphi'(x)\right] + \operatorname{sen}(3x)\left[-\operatorname{sen}(x)\,\varphi(x) + 2\cos(x)\,\varphi'(x) + \operatorname{sen}(x)\,\varphi''(x)\right]$$

$$= \left[3\cos(3x)\cos(x) - \operatorname{sen}(3x)\operatorname{sen}(x)\right]\varphi(x) + \left[3\cos(3x)\operatorname{sen}(x) + 2\cos(x)\operatorname{sen}(3x)\right]\varphi'(x) + \left[\operatorname{sen}(x)\operatorname{sen}(3x)\right]\varphi''(x)$$

$$\boxed{\langle \delta(x) \mid (3\cos(3x)\cos(x) - \operatorname{sen}(3x)\operatorname{sen}(x))\,\varphi(x) + (3\cos(3x)\operatorname{sen}(x) + 2\cos(x)\operatorname{sen}(3x))\,\varphi'(x) + (\operatorname{sen}(x)\operatorname{sen}(3x))\,\varphi''(x)\rangle}$$

$$\langle \delta_{(x)} | (3\cos(3x)\cos(x) - \operatorname{sen}(3x)\operatorname{sen}(x))\varphi_{(x)}\rangle + \langle \delta_{(x)} | (3\cos(3x)\operatorname{sen}(x) + 2\cos(x)\operatorname{sen}(3x))\varphi'_{(x)}\rangle$$

$$+ \langle \delta_{(x)} | (\operatorname{sen}(3x)\operatorname{sen}(x))\varphi''_{(x)}\rangle$$

$$= 3\varphi_{(0)} \;\Rightarrow\; 3\varphi_{(0)} = 3\langle \delta_{(x)} | \varphi_{(x)}\rangle = \langle \delta_{(x)} | (\operatorname{sen}(3x)(\operatorname{sen}(x)\varphi_{(x)})')'\rangle$$

$$= \langle (\operatorname{sen}(x)(\operatorname{sen}(3x)\delta'_{(x)}))' | \varphi_{(x)}\rangle = 3\langle \delta_{(x)} | \varphi_{(x)}\rangle$$

$$\Rightarrow \boxed{\operatorname{sen}(x)(\operatorname{sen}(3x)\delta'_{(x)})' = 3\delta_{(x)}}$$

---

2) (PARCIAL SEP-DIC 2012) (12 Puntos).

Exprese la función generalizada (definida en el espacio de funciones suaves en un entorno de cero)

$$f(x) = (1-x^2)^{1/7}\,\delta''_{(x)}$$

Como combinación lineal de $\delta_{(x)}$; $\delta'_{(x)}$ y $\delta''_{(x)}$.

Solución

$$\langle (1-x^2)^{1/7}\delta''_{(x)} | \varphi_{(x)}\rangle = \langle \delta''_{(x)} | (1-x^2)^{1/7}\varphi_{(x)}\rangle = \langle \delta_{(x)} | ((1-x^2)^{1/7}\varphi_{(x)})''\rangle$$

$$((1-x^2)^{1/7}\varphi_{(x)})'' = \Big[\underbrace{\tfrac{1}{7}(1-x^2)^{-6/7}\cdot(-2x)\cdot\varphi(x)}_{A} + \underbrace{(1-x^2)^{1/7}\cdot\varphi'_{(x)}}_{B}\Big]'$$

$$B \to ((1-x^2)^{1/7}\cdot\varphi'_{(x)})' = \frac{1}{7}(1-x^2)^{-6/7}(-2x)\varphi'_{(x)} + (1-x^2)^{1/7}\varphi''_{(x)}$$

$$A \to \left(\frac{1}{7}(1-x^2)^{-6/7}(-2x)\varphi_{(x)}\right)' = -\frac{6}{49}(1-x^2)^{-13/7}\cdot(4x^2)\varphi_{(x)} - \frac{2}{7}(1-x^2)^{-6/7}\varphi_{(x)} - \frac{2}{7}(1-x^2)^{-6/7}\cdot(x)\varphi'_{(x)}$$

$$= -\frac{24}{49}x^2\cdot(1-x^2)^{-13/7}\cdot\varphi_{(x)} - \frac{2}{7}(1-x^2)^{-6/7}\cdot\varphi_{(x)} - \frac{2}{7}(1-x^2)^{-6/7}\cdot x\cdot\varphi'_{(x)}$$

$$A + B = (1-x^2)^{1/7}\varphi''_{(x)} - \frac{4}{7}(x)(1-x^2)^{-6/7}\varphi'_{(x)} + \left(-\frac{24}{49}x^2(1-x^2)^{-13/7} - \frac{2}{7}(1-x^2)^{-6/7}\right)\varphi_{(x)}$$

$$\langle \delta_{(x)} | ((1-x^2)^{1/7}\varphi_{(x)})''\rangle = \langle \delta_{(x)} | (1-x^2)^{1/7}\varphi''_{(x)} - \frac{4}{7}x\cdot(1-x^2)^{-6/7}\varphi'_{(x)} + \left(-\frac{24}{49}\cdot x^2(1-x^2)^{-13/7} - \frac{2}{7}(1-x^2)^{-6/7}\right)\varphi_{(x)}\rangle$$

$$= \langle \delta(x) \,|\, (1-x^2)^{1/7} \varphi''(x) \rangle - \langle \delta(x) \,|\, \tfrac{4}{7} x (1-x^2)^{-6/7} \varphi'(x) \rangle + \langle \delta(x) \,|\, \left(-\tfrac{24}{49} x^2 (1-x^2)^{-13/7} - \tfrac{2}{7}(1-x^2)^{-6/7} \right) \varphi(x) \rangle$$

$$= \varphi''(0) - \frac{2}{7}\varphi(0)$$

$$\varphi''(0) = \langle \delta''(x) \,|\, \varphi(x) \rangle \quad \wedge \quad -\frac{2}{7}\varphi(0) = -\frac{2}{7}\langle \delta(x) \,|\, \varphi(x) \rangle$$

$$= \langle \delta''(x) - \frac{2}{7}\delta(x) \,|\, \varphi(x) \rangle$$

$$\Rightarrow \langle (1-x^2)^{1/7} \delta''(x) \,|\, \varphi(x) \rangle = \langle \delta''(x) - \frac{2}{7}\delta(x) \,|\, \varphi(x) \rangle$$

$$\Rightarrow \boxed{f(x) = (1-x^2)^{1/7} \delta''(x) = \delta''(x) - \frac{2}{7}\delta(x)}$$

---

3) (PARCIAL ENERO – MARZO 2012) (10 Pts)

Sea $f(x)$ dada por

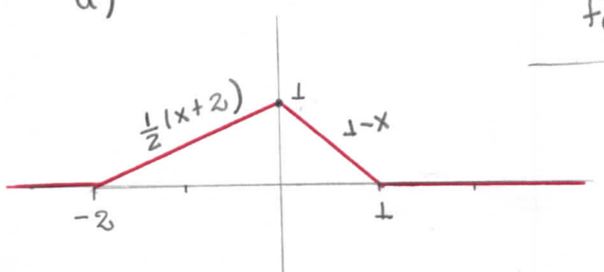
$$f(x) = \begin{cases} \frac{1}{2}(x+2) & ; x \in (-2;0) \\ 1-x & ; x \in [0;1] \\ 0 & ; \text{en cualquier otro } x. \end{cases}$$

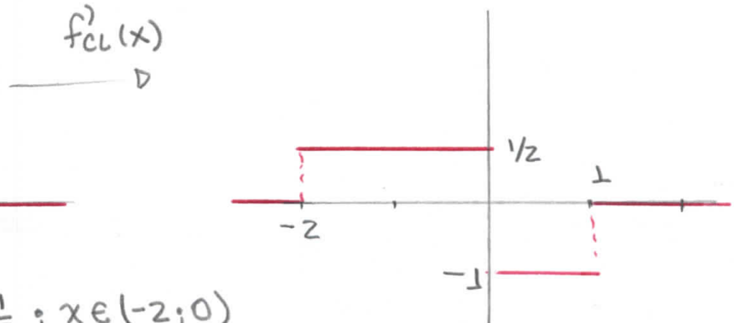
a) Calcule $f''_{gen}(x)$ y $f'_{gen}(x)$.

b) Calcule $I = \int_{-\infty}^{\infty} f(x) \operatorname{sen}(ax)\, dx$; $a \in \mathbb{R}$.

Solución

a)





$$f'_{gen}(x) = f'_{CL}(x) = \begin{cases} \frac{1}{2} ; & x \in (-2;0) \\ -1 ; & x \in (0;1) \\ 0 & \text{en cualquier otro } x \end{cases}$$

$$\boxed{f''_{CL}(x) = 0}$$

$$f''_{gen}(x) = \underbrace{f''_{CL}(x)}_{=0} + \frac{1}{2}\delta_{-2}(x) - \frac{3}{2}\delta(x) + \delta_1(x)$$

$$\Rightarrow \boxed{f''_{gen}(x) = \frac{1}{2}\delta_{-2}(x) - \frac{3}{2}\delta(x) + \delta_1(x).}$$

b) Calcule $I = \int_{-\infty}^{\infty} f(x)\,\mathrm{sen}(ax)\,dx \; ; \; a \in \mathbb{R} \wedge a \neq 0$

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{sen}(ax)\,dx = \int_{-2}^{2} f(x)\,\mathrm{sen}(ax)\,dx$$

Por definición $\langle f_{gut}^{(n)}(x) | \varphi(x) \rangle = (-1)^n \int_{-\infty}^{\infty} f(x)\,\varphi^{(n)}(x)\,dx$

$$\varphi''(x) = \mathrm{sen}(ax) \rightarrow \varphi'(x) = \frac{-1}{a}\cos(ax)$$

$$\langle f_{gut}''(x) | \varphi(x) \rangle = \int_{-\infty}^{\infty} f(x)\,\varphi''(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\mathrm{sen}(ax)\,dx \Rightarrow \boxed{\varphi(x) = -\frac{1}{a^2}\,\mathrm{sen}(ax)}$$

$$\langle f_{gut}''(x) | \varphi(x) \rangle = \langle \tfrac{1}{2}\delta_{-2}(x) - \tfrac{3}{2}\delta(x) + \delta_1(x) \,|\, -\tfrac{1}{a^2}\,\mathrm{sen}(ax) \rangle$$

$$= -\frac{1}{a^2}\left[\frac{1}{2}\langle \delta_{-2}(x) | \mathrm{sen}(ax) \rangle - \frac{3}{2}\underbrace{\langle \delta(x) | \mathrm{sen}(ax) \rangle}_{=0} + \langle \delta_1(x) | \mathrm{sen}(ax) \rangle\right]$$

$$= -\frac{1}{a^2}\left[\frac{1}{2}\,\mathrm{sen}(-2a) + \mathrm{sen}(a)\right]$$

$$\boxed{\mathrm{SEN}(-x) = -\mathrm{SEN}(x)}$$

$$\Rightarrow \langle f_{gut}''(x) | \varphi(x) \rangle = \frac{1}{a^2}\left[\frac{1}{2}\,\mathrm{sen}(2a) - \mathrm{sen}(a)\right]$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\mathrm{sen}(ax)\,dx = \frac{1}{a^2}\left[\frac{1}{2}\,\mathrm{sen}(2a) - \mathrm{sen}(a)\right]}$$

4) (Septiembre – Diciembre 2012).

1. a) 6 Ptos. Halle una función real y continua f que satisfaga

$$\begin{cases} f(x) \equiv 0 \text{ en } -\infty < x < 7 \\ f'''_{gen}(x) = \frac{1}{7}\delta'(x+7) - \frac{8}{7}\delta'(x) + \delta'(x-1) \end{cases}$$

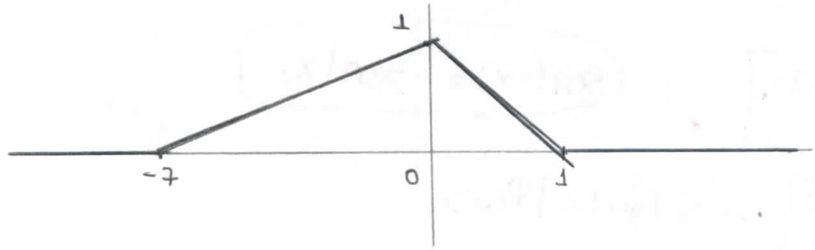
Solución

$$f'''_{gen}(x) = \frac{1}{7}\delta'(x+7) - \frac{8}{7}\delta'(x) + \delta'(x-1).$$

Integramos $f''_{gen}(x) = \frac{1}{7}\delta(x+7) - \frac{8}{7}\delta(x) + \delta(x-1) + C_1$

Integramos $f'_{gen}(x) = \frac{1}{7}H(x+7) - \frac{8}{7}H(x) + H(x-1) + C_1x + C_2$

Integramos $f(x) = \frac{1}{7}(x+7)H(x+7) - \frac{8}{7}xH(x) + (x-1)H(x-1) + C_1\frac{x^2}{2} + C_2x + C_3.$



		-7	0	1
$\frac{1}{7}(x+7)H(x+7)$	—	✓	✓	✓
$-\frac{8}{7}xH(x)$	—	—	✓	✓
$(x-1)H(x-1)$	—	—	—	✓
$C_1\frac{x^2}{2} + C_2x + C_3$	✓	✓	✓	✓

$$f(x) = \frac{1}{7}(x+7)H(x+7) - \frac{8}{7}xH(x) + (x-1)H(x-1)$$

1.b) 6 Ptos.

Calcule el valor de la integral $\int_{-\infty}^{\infty} f(x)\, \mathrm{sen}(\lambda x)\, \partial x$ siendo $\lambda \in \mathbb{R}$.

$$f''_{gen}(x) = \frac{1}{7}\delta(x+7) - \frac{8}{7}\delta(x) + \delta(x-1)$$

$$\langle f^{n}_{gen}(x) | \varphi(x) \rangle = (-1)^n \int_{-\infty}^{\infty} f(x)\, \varphi^{n}(x)\, \partial x$$

$$\varphi''(x) = \mathrm{sen}(\lambda x)$$

$$\langle f''_{gen}(x) | \varphi(x) \rangle = \int_{-\infty}^{\infty} f(x)\, \varphi''(x)\, \partial x = \int_{-\infty}^{\infty} f(x)\, \mathrm{sen}(\lambda x)\, \partial x \qquad \varphi(x) = \frac{-1}{\lambda^2}\mathrm{sen}(\lambda x)$$

$$\langle \frac{1}{7}\delta(x+7) - \frac{8}{7}\delta(x) + \delta(x-1) \,|\, -\frac{1}{\lambda^2}\mathrm{sen}(\lambda x) \rangle$$

$$= -\frac{1}{\lambda^2}\left[\frac{1}{7}\langle \delta(x+7) | \mathrm{sen}(\lambda x) \rangle - \cancel{\frac{8}{7}\langle \delta(x) | \mathrm{sen}(\lambda x) \rangle}_{=0} + \langle \delta(x-1) | \mathrm{sen}(\lambda x) \rangle \right]$$

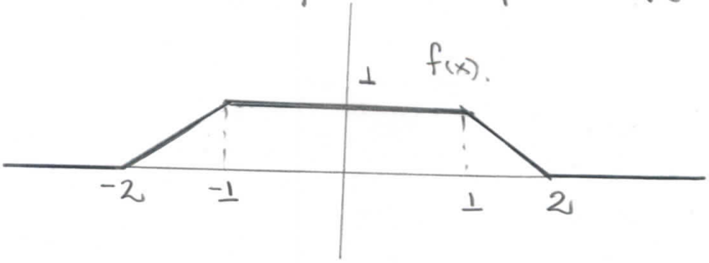
$$= -\frac{1}{\lambda^2}\left[\frac{1}{7}\mathrm{sen}(-7\lambda) + \mathrm{sen}(\lambda) \right] \quad ; \quad \mathrm{sen}(-x) = -\mathrm{sen}(x)$$

$$= \frac{1}{\lambda^2}\left[\frac{1}{7}\mathrm{sen}(7\lambda) - \mathrm{sen}(\lambda) \right] = \langle f''_{gen}(x) | \varphi(x) \rangle$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\, \mathrm{sen}(\lambda x)\, \partial x = \frac{1}{\lambda^2}\left[\frac{1}{7}\mathrm{sen}(7\lambda) - \mathrm{SEN}(\lambda) \right]}$$

5) (Parcial Sep - Dic 2008) 8 Ptos

a) Dada la gráfica encuentre una función que describa el comportamiento de tal gráfica como combinacion lineal de funciones Heaviside y como combinación de funciones pulsos. (2 Ptos) (3 Ptos).



Solución

⇒ Para $x \leq -2$ $f(x) = 0$.

⇒ Para $x > -2$ $f(x) = (x+2)H(x+2)$.

⇒ Para $x \in [-1;1]$; $f(x) \equiv 1 \Rightarrow f(x) = (x+2)H(x+2) - (x+1)H(x+1)$.

⇒ Para $x \in [1;2]$; $f(x) = (x+2)H(x+2) - (x+1)H(x+1) - (x-1)H(x-1)$

⇒ Para $x \geq 2$ $f(x) \equiv 0$.

$$\Rightarrow f(x) = (x+2)H(x+2) - (x+1)H(x+1) - (x-1)H(x-1) + (x-2)H(x-2).$$

$$f(x) = (x+2)\mathbb{1}_{[-2;-1]} + \mathbb{1}_{[-1;1]} + (2-x)\mathbb{1}_{[1;2]}$$

2) Demuestre que $f''_{gen}(x) = \delta(x+2) + \delta(x+1) - \delta(x-1) + \delta(x-2)$. (3 Ptos)

$$f(x) = (x+2)H(x+2) - (x-1)H(x+1) - (x-1)H(x-1) + (x-2)H(x-2)$$

$$f'_{gen}(x) = H(x+2) - H(x+1) - H(x-1) + H(x-2).$$

$$f''_{gen}(x) = \delta(x+2) - \delta(x+1) - \delta(x-1) + \delta(x-2)$$